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# MODIFIED THREE-DIMENSIONAL FORMULATIONS OF BENDING PROBLEMS OF HOMOGENEOUS PLATES AND BEAMS UNDER COMPLEX FIXING CONDITIONS

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*Modified three-dimensional formulations of bending problems of homogeneous elastic plates and beams are considered. Modification of the known three-dimensional formulations reduces to using additional constraints imposed on displacement functions. An advantage of the formulations proposed is that complex fixing conditions of plates and beams can be taken into account.*

**Key words:** *modified theories of plates, finite-element method.*

**Introduction.** The known theories of bending of homogeneous elastic plates and beams are based on a series of hypotheses [1–4]. The use of hypotheses, on the one hand, simplifies the solution of the problems (reduces their dimensionality) and, on the other hand, imposes certain restrictions on displacement, strain, and stress fields, which introduces an unavoidable error in the solutions. Moreover, the existing theories of bending of homogeneous plates and beams fail to take into account complex fixing conditions, for example, a partially clamped edge of a plate. Finite-element three-dimensional models of homogeneous plates and beams allow one to take into account all fixing conditions and obtain grid solutions with a specified accuracy, but these models have large dimensionality.

In the present paper, we consider modified three-dimensional formulations of the problems of bending of homogeneous elastic plates and beams. Modification of the three-dimensional formulations reduces to imposing additional constraints on displacement functions [5, 6]. It should be noted that the use of these constraints in discrete models of plates and beams leads to substantial reduction of their dimensionality. The additional constraints imposed on displacement functions are specified only in a region of a plate (beam) far from the fixed part of the boundary. Thus, no hypotheses are introduced for the displacement, strain, and stress fields in the neighborhood of the fixed edge, i.e, a three-dimensional stress state occurs.

An advantage of the formulations proposed is that the implementation of the finite-element method (FEM) for modified formulations of bending problems of plates and beams requires much less computational effort compared to finite-element implementation for three-dimensional formulations. The formulations proposed describe the three-dimensional stress state in the neighborhood of fixed boundaries of plates and beams, which makes it possible to take into account complex kinematic boundary conditions.

**1. Modified Three-Dimensional Formulations of Bending Problems of Plates and Beams.** 1.1. Let an isotropic homogeneous linear-elastic thin plate occupy a domain $V$ in the Cartesian coordinate system $xyz$. The middle plane of the plate coincides with the $xOy$ plane. The displacements, strains, and stresses of the plate satisfy the Cauchy relations and Hooke's law [1]. The plate is loaded by surface forces $q_z = q_z(x, y)$ and fixed along the boundaries $S_\alpha$: $u = v = w = 0$, where $u$, $v$, and $w$ are the displacement functions and $\alpha = 1, \ldots, M$ ($M$ is the total number of boundaries). We denote a subdomain in the neighborhood of the boundary $S_\alpha$ by $V_\alpha$. The subdomain $V_\alpha$ can be considered as a set of spheres of radius $R_\alpha \geqslant C_\alpha$ ($C_\alpha$ are certain numbers) whose centers lie at the boundary $S_\alpha$. As the calculations show, it is expedient to use the values $C_\alpha \geqslant 2h_0$ ($h_0$ is the plate thickness) and choose the shape of the domain $V_\alpha$ for convenience reasons. Let $S_r = \sum S_\alpha$ and $V_0 = V - V_r$, where

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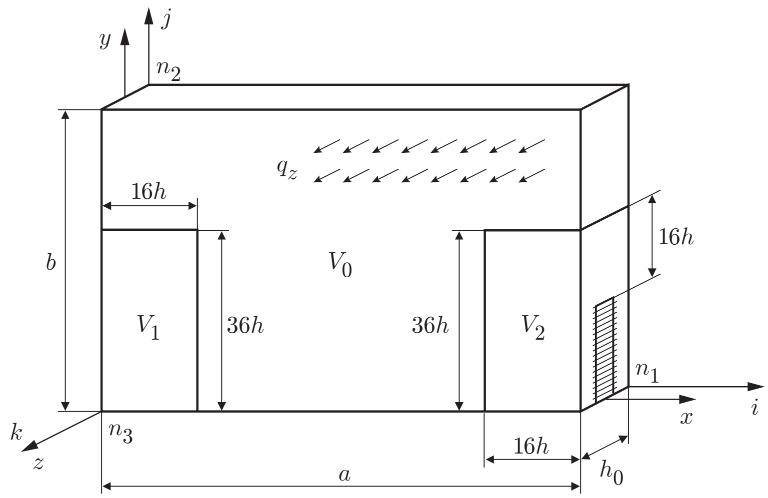


Fig. 1

$V_r = \sum V_\alpha$ ($V_0 \neq \varnothing$). The modified displacement formulation of the three-dimensional elastic problem for the plate $V$ comprises the following equations:

$$\text{in } V, \qquad A(\boldsymbol{u}) = \boldsymbol{p}; \tag{1}$$

$$\text{on } S_q, \quad B(\boldsymbol{u}) = \boldsymbol{q}, \qquad \text{on } S_r, \quad u = v = w = 0; \tag{2}$$

$$\text{in } V_0, \qquad \begin{matrix} u(x,y,z) = -u(x,y,-z), & v(x,y,z) = -v(x,y,-z), \\ u(x,y,0) = v(x,y,0) = 0, & w(x,y,z) = w(x,y,0). \end{matrix} \tag{3}$$

Here $\boldsymbol{u} = \{u, v, w\}^{\rm t}$, $A$ is the equilibrium-equation operator, $\boldsymbol{p} = \{0, 0, 0\}^{\rm t}$ is the vector of body forces, $B$ is the operator of the static boundary conditions, $\boldsymbol{q} = \{0, 0, q_z\}^{\rm t}$ is the surface-force vector, $S_q$ is the boundary at which the load $\boldsymbol{q}$ is specified, $S = S_r + S_q$ is the boundary of the domain $V$, and $w$ is the deflection of the plate.

The calculations show that, for three-dimensional homogeneous plates, equalities (3) are satisfied approximately, i.e.,

$$\text{in } V_0, \qquad \begin{matrix} |u^+ - u^-| < \varepsilon_1, & |v^+ - v^-| < \varepsilon_2, \\ |w^+ - w^-| < \varepsilon_3, & |u(x,y,0)| < \varepsilon_4, \quad |v(x,y,0)| < \varepsilon_4. \end{matrix} \tag{4}$$

Here $\varepsilon_1, \ldots, \varepsilon_4$ are small numbers, $u^+ = |u(x,y,z)|$, $u^- = |u(x,y,-z)|$, $v^+ = |v(x,y,z)|$, $v^- = |v(x,y,-z)|$, $w^+ = |w(x,y,z)|$, and $w^- = |w(x,y,0)|$.

By virtue of (4), the solution obtained under conditions (3) differs from the exact solution with an error $\delta$. This error depends on the dimensions of the subdomains $V_\alpha$ determined numerically (for specific fixing conditions of the plate) under the condition that the error of the solution $\delta$ does not exceed a certain value. The calculations show that the error $\delta$ tends to zero as the geometrical dimensions of the subdomains $V_\alpha$ increase.

Thus, the modified formulation of the three-dimensional elastic problem of a homogeneous thin plate differs from the three-dimensional formulation (1), (2) by additional constraints (3) imposed on the displacement functions of the plate. In the domains $V_\alpha$, no hypotheses are introduced for the displacement, strain, and stress fields, i.e., a three-dimensional stress state occurs in $V_r = \sum V_\alpha$.

We consider the modified three-dimensional finite-element model of the plate $V$. First, we construct a discrete (basic) model of the plate using the known formulation of the three-dimensional elastic problem, i.e., relations (1) and (2). The basic model consists of the first-order finite elements $V_e^h$ shaped as a cube with a side $h$ ($e = 1, \ldots, N$ is the total number of finite elements [7, 8]). The nodal unknowns of the elements $V_e^h$ are the displacements $u$, $v$, and $w$. The basic discretization of the plate yields a three-dimensional nodal grid $V_h$ with the cell size $h$ along the axes $x$, $y$, and $z$ and dimensionality $n_1 \times n_2 \times n_3$, where $n_3 = 2k_0 + 1$ and $k_0$ is an integer. For the nodes of the grid $V_h$, we introduce the integer coordinate system $ijk$ (Fig. 1). In Fig. 1, $n_1 = 81$, $n_2 = 51$, $n_3 = 9$, $a = 80h$, $b = 50h$, $h_0 = 8h$, and $k_0 = 4$. Conditions (3) for the nodes of the grid $V_h$ that enter the domain $V_0$ imply the following relations:

$$\forall(i,j,k)\in V_0,\qquad \begin{array}{l} u(i,j,k)=-u(i,j,n_3-k+1),\\ v(i,j,k)=-v(i,j,n_3-k+1),\qquad k=1,\dots,k_0,\\ u(i,j,k_0+1)=v(i,j,k_0+1)=0,\\ w(i,j,k)=w(i,j,k_0+1),\qquad k=1,\dots,n_3,\quad k\neq k_0+1. \end{array} \tag{5}$$

Here $u(i,j,k)$, $v(i,j,k)$, and $w(i,j,k)$ are the displacements $u$, $v$, and $w$ of the node $(i,j,k)$ of the grid $V_h$ whose coordinates are specified in the integer coordinate system $ijk$.

We write the potential energy $\Pi$ of the basic model of the plate in the matrix form [7]

$$\Pi(\{\delta_h\})=\{\delta_h\}^{\mathrm{t}}[K_h]\{\delta_h\}/2-\{\delta_h\}^{\mathrm{t}}\{P_h\}, \tag{6}$$

where $[K_h]$ is the stiffness matrix of the basic model and $\{P_h\}$ and $\{\delta_h\}$ are the vectors of nodal forces and nodal unknowns.

We write the vector $\{\delta_h\}$ as

$$\{\delta_h\}=\{\{\delta_+^{uv}\}\ \{\delta_-^{uv}\}\ \{\delta_0^{uv}\}\ \{\delta_0^{w}\}\ \{\delta^{w}\}\ \{\delta\}\}^{\mathrm{t}}, \tag{7}$$

where $\{\delta_-^{uv}\}$, $\{\delta_+^{uv}\}$, and $\{\delta_0^{uv}\}$ are the vectors of the displacements $u$ and $v$ of the nodes of the grid $V_h$ which lie in the domains $V_0^1=\{x,y,z\in V_0,\ z<0\}$ and $V_0^2=\{x,y,z\in V_0,\ z>0\}$ and in the plane $S_0=\{x,y,z\in V_0,\ z=0\}$, respectively, $\{\delta_0^w\}$ and $\{\delta^w\}$ are the vectors of the displacements $w$ of the nodes of the grid $V_h$ located in the plane $S_0$ and in the domain $V_0^3=\{x,y,z\in V_0,\ z\neq 0\}$, and $\{\delta\}$ is the vector of the remaining nodal unknowns of the basic model of the plate.

Writing equalities (5) for the nodes of the grid $V_h$, we obtain

$$\{\delta_0^{uv}\}=0,\qquad \{\delta_-^{uv}\}=[A]\{\delta_+^{uv}\},\qquad \{\delta^w\}=[B]\{\delta_0^w\}, \tag{8}$$

where $[A]$ is a square matrix whose elements are equal to either zero or $-1$ and $[B]$ is a rectangular Boolean matrix.

With allowance for $\{\delta_0^{uv}\}=0$, vector (7) becomes

$$\{\delta_h\}=\{\{\delta_+^{uv}\}\ \{\delta_-^{uv}\}\ \{\delta_0^{w}\}\ \{\delta^{w}\}\ \{\delta\}\}^{\mathrm{t}}. \tag{9}$$

We introduce the vector of nodal unknowns of the modified three-dimensional discrete model of the plate

$$\{\delta_0\}=\{\{\delta_+^{uv}\}\ \{\delta_0^{w}\}\ \{\delta\}\}^{\mathrm{t}}. \tag{10}$$

Using (8) and (9), we obtain the relation between the vectors $\{\delta_h\}$ and $\{\delta_0\}$

$$\{\delta_h\}=[K]\{\delta_0\}, \tag{11}$$

where $[K]$ is the rectangular matrix

$$[K]=\begin{bmatrix} [E_1] & 0 & 0\\ [A] & 0 & 0\\ 0 & [E_2] & 0\\ 0 & [B] & 0\\ 0 & 0 & [E_3] \end{bmatrix}$$

and $[E_k]$ is the unit matrix ($k=1,2,3$). Substituting (11) into (6), from the condition $\partial\Pi/\partial\{\delta_0\}=0$, we obtain the system of equations $[K_0]\{\delta_0\}=\{P_0\}$, where $[K_0]=[K]^{\mathrm{t}}[K_h][K]$ is the stiffness matrix and $\{P_0\}=[K]^{\mathrm{t}}\{P_h\}$ is the vector of nodal forces of the modified three-dimensional discrete model of the plate.

A comparison of (7) with (10) shows that the dimensionality of the vector $\{\delta_0\}$ is smaller than that of the vector $\{\delta_h\}$. Thus, the dimensionality of the modified discrete model of the plate is smaller than that of the discrete model based on the three-dimensional formulation [Eqs. (1) and (2)].

1.2. Let us consider an isotropic homogeneous linear-elastic beam that occupies the domain $V$ in the Cartesian coordinate system $xyz$. The beam axis coincides with the $Ox$ axis and the planes $xOy$ and $xOz$ are the horizontal and vertical planes of the geometrical symmetry of the beam, respectively. The displacements, strains, and stresses of the beam satisfy the Cauchy relations and Hooke's law [1]. The beam is loaded by the surface forces $q_z$ such that $q_z(x,y,z)=q_z(x,-y,z)$, i.e., the function $q_z$ is symmetric relative to the $zOx$ plane. The beam is fixed at the boundaries $S_\alpha$: $u=v=w=0$, where $\alpha=1,\dots,M$ ($M$ is the number of sectors of the boundary over which the beam is fixed). We denote the subdomains that contain the boundary $S_\alpha$ by $V_\alpha^1$ and $V_\alpha^2$, where $V_\alpha^1\subset V_\alpha^2$. The domain $V_\alpha^1$ (or $V_\alpha^2$) can be considered as a set of spheres of radius $R_\alpha^1\geqslant C_\alpha^1$ (or $R_\alpha^2\geqslant C_\alpha^2$) whose

TABLE 1

| $i$ | $j = 31$ | | $j = 51$ | |
|---|---|---|---|---|
| | $w_0$ | $w_h$ | $w_0$ | $w_h$ |
| 11 | 152.415 | 152.084 | 373.795 | 372.595 |
| 21 | 214.764 | 213.642 | 453.960 | 451.957 |
| 31 | 265.052 | 262.759 | 529.946 | 527.048 |
| 41 | 297.266 | 294.054 | 591.584 | 588.137 |
| 51 | 304.433 | 301.308 | 624.574 | 621.178 |
| 61 | 282.470 | 280.308 | 622.324 | 619.704 |
| 71 | 232.827 | 231.874 | 589.996 | 588.683 |
| 81 | 164.026 | 164.114 | 541.301 | 541.593 |

TABLE 2

| $x$ | $z = -0.5h$ | | $z = -1.5h$ | | $z = -2.5h$ | | $z = -3.5h$ | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma_0$ | $\sigma_h$ | $\sigma_0$ | $\sigma_h$ | $\sigma_0$ | $\sigma_h$ | $\sigma_0$ | $\sigma_h$ |
| $0.5h$ | 5.1653 | 5.1588 | 10.1284 | 10.1040 | 15.2231 | 15.2028 | 11.6919 | 11.6910 |
| $4.5h$ | 4.2264 | 4.2106 | 3.1628 | 3.1513 | 2.3911 | 2.3847 | 1.9803 | 1.9785 |
| $9.5h$ | 3.1692 | 3.1439 | 2.2933 | 2.2757 | 1.5017 | 1.4917 | 0.8984 | 0.8948 |
| $29.5h$ | 2.0083 | 1.9518 | 1.4318 | 1.3902 | 0.8678 | 0.8424 | 0.3351 | 0.3264 |
| $49.5h$ | 2.2095 | 2.1251 | 1.5709 | 1.5082 | 0.9530 | 0.9145 | 0.3779 | 0.3655 |
| $69.5h$ | 4.4536 | 4.4065 | 3.2092 | 3.1765 | 2.1327 | 2.1132 | 1.3613 | 1.3504 |
| $74.5h$ | 6.2365 | 6.2096 | 4.5684 | 4.5482 | 3.3622 | 3.3496 | 2.7161 | 2.7095 |
| $79.5h$ | 9.9274 | 9.9152 | 16.6218 | 16.5722 | 26.9340 | 26.8874 | 22.8005 | 22.7953 |

centers lie at the boundary $S_\alpha$. In practice, it is recommended to use the values $C_\alpha^1 \geqslant 2h_0$ and $C_\alpha^2 \geqslant 4h_0$ ($h_0$ is the characteristic cross-sectional size of the beam). In this case, the shape of the domain $V_\alpha^1$ ($V_\alpha^2$) is chosen for convenience reasons. We introduce the notation $S_r = \sum S_\alpha$, $V_0^1 = V - V_r^1$, and $V_0^2 = V - V_r^2$, where $V_r^1 = \sum V_\alpha^1$ and $V_r^2 = \sum V_\alpha^2$ ($V_0^1 \neq \varnothing$ and $V_0^2 \neq \varnothing$).

The modified displacement formulation of the three-dimensional elastic problem of a homogeneous beam comprises the following equations: equations of equilibrium, static and kinematic boundary conditions [equations of the form (1) and (2)], and additional conditions for beam displacements:

$$\text{in } V_0^1, \quad \begin{array}{l} u(x,y,0) = 0, \qquad v(x,y,0) = 0, \\ u(x,y,z) = -u(x,y,-z), \qquad v(x,y,z) = -v(x,y,-z); \end{array} \tag{12}$$

$$\text{in } V_0^2, \qquad w(x,y,z) = w(x,y,0), \quad v(x,y,z) = 0 \tag{13}$$

($w$ is the beam deflection).

It should be noted that a three-dimensional stress state occurs in the neighborhood of the fixed boundaries of the beam $S_\alpha$ (i.e., in $V_r^1$). Satisfying equalities (12) and (13) for the discrete basic model of the beam in a similar manner as in Sec. 1.1, we obtain a modified three-dimensional discrete model of the beam.

**2. Results of Numerical Experiments.** 2.1. We consider a $80h \times 50h \times h_0$ homogeneous isotropic plate ($h_0$ is the plate thickness) (Fig. 1). The plate is fixed at the boundary $S_r = S_1 + S_2$, where $S_1 = \{x = 0,\ 0 \leqslant y \leqslant 20h,\ -2h \leqslant z \leqslant 2h\}$ and $S_2 = \{x = a,\ 0 \leqslant y \leqslant 20h,\ -2h \leqslant z \leqslant 2h\}$, i.e., the plate is partly clamped for $x = 0$ and $x = a$. In Fig. 1, the boundary $S_2$ is shaded. We assume that $V_1 = V_2 = 16h \times 36h \times h_0$ (i.e., $C_1 = C_2 \geqslant 2h_0$) for the boundaries $S_1$ and $S_2$. The basic model of the plate (based on the equations of the three-dimensional elastic problem) consists of the first-order finite elements $V_e^h$ shaped as a cube with a side $h$ and generates an $81 \times 51 \times 9$ nodal grid $V_h$. To determine the nodes of the grid $V_h$, we introduce the integer coordinate system $ijk$ (Fig. 1). At the nodes $(i,j,9)$ of the grid $V_h$, the plate is loaded by the forces $q_z = 1.83$ ($i = 41, 36, 41, 46, \ldots, 76$; $j = 41$, $46$). Young's modulus of the plate is $E = 1$, and Poisson's ratio is $\nu = 0.3$; $h = 0.5$.

An analysis of results shows that the solution $w_h$ (deflection of the plate) for the discrete modified model of the plate differs from the solution of the basic model $w_0$ in the region of the maximum (in absolute value) displacements by no more than 0.54%. The values of the deflections $w_0$ and $w_h$ ($k = 9$) are given in Table 1. Table 2 ($y = 20.5h$) compares the equivalent stresses $\sigma_h$ (for the modified model) with $\sigma_0$ (for the basic model) calculated at the centroid of the elements $V_e^h$ according to the fourth strength theory [9]. The maximum stresses $\sigma_h$ occur in the neighborhood of the fixed boundary of the plate $S_2$ and differ from $\sigma_0$ by no more than 0.18%.

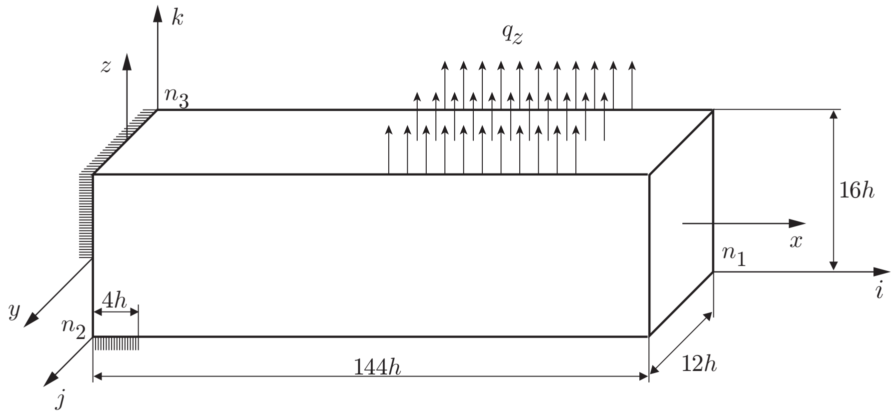


Fig. 2

TABLE 3

| $i$ | $w_0$ | $w_h$ | $i$ | $w_0$ | $w_h$ |
|---|---|---|---|---|---|
| 13 | 13.882 | 13.900 | 109 | 543.205 | 531.344 |
| 25 | 48.520 | 48.532 | 121 | 620.955 | 606.331 |
| 61 | 235.163 | 232.414 | 145 | 776.454 | 756.306 |
| 85 | 388.089 | 381.138 | | | |

TABLE 4

| $x$ | $z=-7.5h$ | | $z=-6.5h$ | | $z=6.5h$ | | $z=7.5h$ | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma_0$ | $\sigma_h$ | $\sigma_0$ | $\sigma_h$ | $\sigma_0$ | $\sigma_h$ | $\sigma_0$ | $\sigma_h$ |
| $0.5h$ | 2.0007 | 1.9778 | 0.9427 | 0.9329 | 1.9559 | 1.9735 | 2.5064 | 2.5277 |
| $3.5h$ | 5.2597 | 5.2167 | 1.7344 | 1.7178 | 2.0127 | 2.0293 | 2.3123 | 2.3281 |
| $4.5h$ | 3.8302 | 3.8006 | 2.3034 | 2.2835 | 2.0013 | 2.0177 | 2.3144 | 2.3301 |
| $6.5h$ | 2.7585 | 2.7390 | 2.2438 | 2.2259 | 2.0025 | 2.0188 | 2.3269 | 2.3428 |
| $8.5h$ | 2.4592 | 2.4421 | 2.1151 | 2.0983 | 1.9964 | 2.0127 | 2.3174 | 2.3335 |
| $12.5h$ | 2.2239 | 2.2076 | 1.9403 | 1.9239 | 1.9226 | 1.9388 | 2.2243 | 2.2404 |
| $24.5h$ | 1.7294 | 1.7146 | 1.5032 | 1.4887 | 1.5073 | 1.5237 | 1.7355 | 1.7506 |
| $44.5h$ | 0.9363 | 0.6817 | 0.8189 | 0.6051 | 0.8072 | 0.7289 | 0.9472 | 0.8332 |

The basic model of the plate contains 110,907 nodal unknowns, and the band width of the finite-element system of equations (SE) is 1410. For the modified model, the number of unknowns is 63,243; the band width is equal to 1191 and occupies half as much computer memory as that of the basic model. The FEM implementation time is almost three times shorter compared to the basic model.

2.2. We consider a $144h \times 12h \times h_0$ homogeneous isotropic prismatic beam ($h_0$ is the cross-sectional height of the beam) (Fig. 2). The beam is fixed at the boundary $S_r = S_1$, where $S_1 = \{x = 0,\ -6h \leqslant y \leqslant 6h,\ 0 \leqslant z \leqslant 8h\} \cup \{0 \leqslant x \leqslant 4h,\ -6h \leqslant y \leqslant 6h,\ z = -8h\}$, i.e., the beam is partly fixed at the left end and horizontal support $z = -8h$ for $x = 0$. In Fig. 2, the fixed region is shaded. The basic model of the beam consists of the first-order elements $V_e^h$ shaped as a cube with a side $h$ and generates a $145 \times 13 \times 17$ nodal grid $V_h$. To determine the nodes of the grid $V_h$, we introduce the integer coordinate system $ijk$ as is shown in Fig. 2. For the beam considered, we assume that $V_1^1 = \{x, y, z \in V,\ 0 \leqslant x \leqslant 36h\}$ and $V_1^2 = \{x, y, z \in V,\ 0 \leqslant x \leqslant 68h\}$, i.e., conditions (12) hold for $x \geqslant 2h_0 + b_0$ ($C_1^1 \geqslant 2h_0$) and conditions (13) hold for $x \geqslant 4h_0 + b_0$ ($C_1^2 \geqslant 4h_0$); $b_0 = 4h$. At the nodes of the grid $V_h$ with the coordinates $(i, j, 17)$, the beam is loaded by the forces $q_z = 0.173$ [$i = 37 + 6(k-1)$, $k = 1, \ldots, 11$, and $j = 2, 7, 12$]. Young's modulus of the beam is $E = 1$, $\nu = 0.3$, and $h = 0.5$.

An analysis of results shows that the grid displacements $w_h$ (deflection of the beam) of the discrete modified model differs from the displacements of the basic model $w_0$ by no more than 2.6%. The displacements $w_0$ and $w_h$ ($j = 7$, $k = 17$) are listed in Table 3. Table 4 ($y = 5.5h$) compares the equivalent stresses $\sigma_h$ (for the modified model) and $\sigma_0$ (for the basic model) calculated at the centroids of the elements $V_e^h$ according to the fourth strength theory. The maximum stresses $\sigma_h$ occur near the clamped end of the beam and differ from $\sigma_0$ by no more than 0.8%. The basic model contains 95,628 nodal unknowns, and the FEM SE band width is 708. For the modified discrete model, the number of unknowns is 37,834, the FEM SU band width is 748, the computer memory required to store the finite-element equations is reduced by a factor of 2.4, and the FEM implementation time is reduced by a factor of 2.4 compared to the basic model.

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